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# CONCERNING THE $\ell^p$-CONJECTURE FOR DISCRETE SEMIGROUPS

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Abstract. For $2 < p < \infty$, it is well-known that if $\mathcal{G}$ is a discrete group, then the convolution product $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{G})$ if and only if $\mathcal{G}$ is finite. Here, we present an improvement of this result for an arbitrary discrete semigroup in terms of semifinite semigroups.

## 1. Introduction

The $L^p$-conjecture was first formulated for a locally compact group $\mathcal{G}$ by Rajagopalan in his Ph.D. thesis in 1963 as follows: for $1 < p < \infty$, the convolution product $\phi * \psi$ exists and belongs to $L^p(\mathcal{G})$ for all $\phi, \psi \in L^p(\mathcal{G})$ if and only if $\mathcal{G}$ is compact. However, the first result related to this conjecture is due to Zelazko [**19**] and Urbanik [**18**] in 1961; they proved that the conjecture is true for all locally compact abelian groups. The truth of the conjecture has been established for $p > 2$ by Zelazko [**20**] and Rajagopalan [**12**] independently; see also Rajagopalan's works [**11**] for the case where $p \geq 2$ and $\mathcal{G}$ is discrete, [**12**] for the case where $p = 2$ and $\mathcal{G}$ is totally disconnected, and [**13**] for the case where $p > 1$ and $\mathcal{G}$ is either nilpotent or a semidirect product of two locally compact groups. In the joint work [**14**], they showed that the conjecture is true for $p > 1$ and amenable groups; this result can be also found in Greenleaf's book [**5**]. Rickert [**16**] confirmed the conjecture for $p = 2$. For related results on the subject see also Crombez [**2**] and [**3**], Gaudet and Gamlen [**4**], Johnson [**7**], Kunze and Stein [**8**], Lohoue [**9**], Milnes [**10**], Rickert [**15**], and Zelazko [**21**]. Finally, in 1990, Saeki [**17**] gave an affirmative answer to the conjecture by a completely self-contained proof.

In [**1**], it was considered only the property that $\phi * \psi$ exists for all $\phi, \psi \in L^p(\mathcal{G})$ of a locally compact group $\mathcal{G}$ and proved that for $2 < p < \infty$, $\phi * \psi$ exists for all $\phi, \psi \in L^p(\mathcal{G})$ if and only if $\mathcal{G}$ is compact. For a discrete group $\mathcal{G}$ and $2 < p < \infty$, it follows that $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{G})$ if and only if $\mathcal{G}$ is finite; moreover,

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$\ell^p(\mathcal{G}) \subseteq \ell^2(\mathcal{G})$ for $1 \leq p \leq 2$, and hence $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{G})$ by the Holder inequality.

Our aim in this work is to consider this problem in the setting of discrete semigroups. In order to this end, we first introduce and study a class of semigroups called semifinite semigroups. Next, for $2 < p < \infty$, we obtain a necessary and sufficient condition for that $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$ in terms of this class of semigroups. However, for $1 \leq p \leq 2$, we are not able to characterize semigroups $\mathcal{S}$ for which $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$; that is, the following question remains open.

**Question.** Let $1 < p \leq 2$. For which semigroups $S$, the convolution product $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$?

## 2. Semifinite semigroups

In this section, we introduce and study a large class of semigroups.

**Definitions 2.1.** We say that a semigroup $\mathcal{S}$ is *left semifinite* if the set $\{s \in \mathcal{S} : s^{-1}t \neq \emptyset\}$ is finite for all $t \in \mathcal{S}$, where $s^{-1}t = \{r \in \mathcal{S} : sr = t\}$; similarly, we say that $\mathcal{S}$ is *right semifinite* if the set $\{s \in \mathcal{S} : st^{-1} \neq \emptyset\}$ is finite for all $t \in \mathcal{S}$, where $st^{-1} = \{r \in \mathcal{S} : s = rt\}$. A left and right semifinite semigroup is simply said to be *semifinite semigroup.*

Let us point out that a left semifinite semigroup is not necessarily right semifinite; for example consider left zero semigroups. Similarly, a left semifinite semigroup is not necessarily left weakly cancellative; recall that $\mathcal{S}$ is called left (resp. right) weakly cancellative if $s^{-1}t$ (resp. $st^{-1}$) is finite for all $s, t \in \mathcal{S}$. Furthermore, a left or right weakly cancellative semigroup is not necessarily left semifinite.

Our next result gives the only direct relation between these concepts which is needed in the sequel.

**Proposition 2.2.** *Any left semifinite semigroup is right weakly cancellative.*

*Proof.* Let $\mathcal{S}$ be a left semifinite semigroup. Then $tr^{-1} \subseteq \{s \in \mathcal{S} : s^{-1}t \neq \emptyset\}$ for all $t, r \in \mathcal{S}$. In particular, $tr^{-1}$ is finite for all $t, r \in \mathcal{S}$ as required. □

Proposition 2.2 and its dual yield the following result.

**Corollary 2.3.** *Any semifinite semigroup is weakly cancellative; i.e., left and right weakly cancellative.*

Let us remark that any infinite group is cancellative and of course weakly cancellative, but not semifinite. So, the converse of Corollary 2.3 is not true. However, we have the following result which describes the interaction between these concepts.

**Theorem 2.4.** *Let $\mathcal{S}$ be a semigroup. Then the following assertions are equivalent.*

***(a)*** *$\mathcal{S}$ is left weakly cancellative and left semifinite.*
***(b)*** *$\mathcal{S}$ is right weakly cancellative and right semifinite.*
***(c)*** *$\mathcal{S}$ is semifinite.*

*Proof.* It is sufficient to show that (a) implies (b). Suppose that (a) holds. By Proposition 2.2, $\mathcal{S}$ is right weakly cancellative. So, we only need to show that $\mathcal{S}$ is right semifinite; suppose on the contrary that there exists $s \in \mathcal{S}$ and an infinite subset $T$ of $\mathcal{S}$ such that $st^{-1} \neq \emptyset$ for all $t \in T$. Since $\mathcal{S}$ is left weakly cancellative, it follows that

$$\bigcap\{st^{-1} : t \in T_0\} = \emptyset$$

for all infinite subsets $T_0$ of $T$. By induction, there exists a sequence $(t_n)$ of distinct elements of $\mathcal{S}$ such that $st_{n+1}^{-1} \nsubseteq st_1^{-1} \cup ... \cup st_n^{-1}$ for all $n \geq 1$. So, we may find an infinite subset $\{r_n : n \geq 1\}$ of $\mathcal{S}$ such that $r_n \in st_n^{-1}$. Therefore, $t_n \in r_n^{-1}s$, and thus the set $\{r \in \mathcal{S} : r^{-1}s \neq \emptyset\}$ is infinite; this is a contradiction. □

We now give some examples of semifinite semigroups.

**Example 2.5.** (a) Let $\mathcal{S}$ be the semigroup $[0,1]$ with the operation $xy = \min\{x+y, 1\}$ for all $x, y \in [0,1]$. Then $\mathcal{S}$ is not semifinite; indeed $y^{-1}1 = [1-y, 1]$ for all $y \in \mathcal{S}$. Also, for every $x < 1$, we get $y^{-1}x = \{x-y\}$ for $y \leq x$ and $y^{-1}x = \emptyset$ otherwise.

(b) Let $X$ be an arbitrary set, and let $\mathcal{S}$ be the power set $\mathcal{P}(X)$ of $X$ endowed with the union operation $\cup$. Then for every $A, B \in \mathcal{P}(X)$, $A^{-1}B \neq \emptyset$ if and only if $A \subseteq B$; in this case $A^{-1}B = \{ (B \setminus A) \cup C : \ C \subseteq A \}$. Hence $\mathcal{S}$ is semifinite only if $X$ is finite. But the uncountable subsemigroup $\mathcal{F}(X)$ of $\mathcal{S}$ consisting of all finite subsets of $\mathcal{S}$ is semifinite.

(c) Let $\mathcal{S}$ be the semigroup consisting of all $2 \times 2$ matrices

$$\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} \quad (x, y \in \mathbb{R}\ ,\ x, y > 0)$$

with the matrix multiplication. Then

$$\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}^{-1} \begin{bmatrix} z & 0 \\ r & 1 \end{bmatrix} \neq \emptyset \quad \text{if and only if} \qquad rx > yz$$

It follows that $\mathcal{S}$ is not left semifinite; similarly, $\mathcal{S}$ is not right semifinite.

Now, we consider the subsemigroup $\mathcal{T}$ of $\mathcal{S}$ consisting of all of $2 \times 2$ matrices in $\mathcal{S}$ with entries in $\mathbb{N}$. Then $\mathcal{T}$ is left semifinite and left weakly cancellative. Now, apply Theorem 2.4 to conclude that $\mathcal{T}$ is semifinite.

## 3. Existence of convolution products

Let $\mathcal{S}$ be a discrete semigroup and $1 \leq p < \infty$. As usual, let $\ell^p(\mathcal{S})$ denote the space of all complex-valued functions on $\mathcal{S}$ with $\sum_{s \in \mathcal{S}} |\phi(s)|^p < \infty$. For functions $\phi$ and $\psi$ on $\mathcal{S}$, define

$$(\phi * \psi)(x) := \sum_{s,t \in \mathcal{S}, x = st} \phi(s)\ \psi(t)$$

at each point $x \in \mathcal{S}^2$ for which this makes sense, and $(\phi * \psi)(x) := 0$ for all $x \in \mathcal{S} \setminus \mathcal{S}^2$, where $\mathcal{S}^2 := \{st : s, t \in \mathcal{S}\}$. We say that $\phi * \psi$ *exists* if $(\phi * \psi)(x)$ exists for all $x \in \mathcal{S}$; in this case, $\phi * \psi$ is called the convolution product of $\phi$ and $\psi$.

It is well-known that $\phi * \psi$ exists and belongs to $\ell^1(\mathcal{S})$ for all $\phi, \psi \in \ell^1(\mathcal{S})$. In this section, we give a description for the existence of $\phi * \psi$ for all $\phi, \psi \in \ell^p(\mathcal{S})$. First, we need a lemma.

**Lemma 3.1.** *Let $\mathcal{S}$ be a semigroup and $1 < p < \infty$. If $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$, then $\mathcal{S}$ is weakly cancellative.*

*Proof.* We show that $\mathcal{S}$ is left weakly cancellative; a similar argument implies that $\mathcal{S}$ is right weakly cancellative. Suppose on the contrary that there exist $s_0, x_0 \in \mathcal{S}$ such that $s_0^{-1}x_0$ is an infinite subset of $\mathcal{S}$. Choose a sequence $(t_n)$ of distinct elements in $s_0^{-1}x_0$. Define

$$\phi(s) = \begin{cases} n^{-1} & \text{if } s = t_n \\ 1 & \text{if } s = s_0 \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\phi \in \ell^p(\mathcal{S})$. Since $x_0 \in \mathcal{S}^2$, it follows that

$$(\phi * \phi)(x_0) = \sum_{s,t \in \mathcal{S}, x_0 = st} \phi(s)\ \phi(t) \geq \sum_{t \in \mathcal{S}, x_0 = s_0 t} \phi(s_0)\ \phi(t) \geq \sum_{n=1}^{\infty} \phi(t_n);$$

hence $(\phi * \phi)(x_0) = \infty$ which is a contradiction. $\square$

We now are ready to state and prove the main result of this paper.

**Theorem 3.2.** *Let $\mathcal{S}$ be a semigroup and $2 < p < \infty$. Then $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$ if and only if $\mathcal{S}$ is semifinite.*

*Proof.* Suppose that $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$. In view of Lemma 3.1, $\mathcal{S}$ is weakly cancellative. To prove that $\mathcal{S}$ is semifinite, by Proposition 2.4, we only need to show that $\mathcal{S}$ is left semifinite. For this end, we suppose on the contrary that there is $x \in \mathcal{S}$ such that

$$X_x := \left\{ s \in \mathcal{S} :\ s^{-1}x \neq \emptyset \right\}$$

is an infinite subset of $\mathcal{S}$. Choose $s_1 \in X_x$. Then there exists $s_2 \in X_x \setminus (\{s_1\} \cup s_1^{-1}x)$ such that

$$s_2^{-1}x \nsubseteq \{s_1\} \cup s_1^{-1}x.$$

Indeed, if $s^{-1}x \subseteq \{s_1\} \cup s_1^{-1}x$ for all $s \in X_x \setminus (\{s_1\} \cup s_1^{-1}x)$, then the set

$$\left\{ s^{-1}x : s \in X_x \setminus (\{s_1\} \cup s_1^{-1}x\ ) \right\}$$

is finite. This together with that $X_x \setminus (\{s_1\} \cup s_1^{-1}x\ )$ is infinite show that there is an infinite subset $X$ of $X_x \setminus (\{s_1\} \cup s_1^{-1}x\ )$ such that $s^{-1}x = t^{-1}x$ for all $s, t \in X$; in particular,

$$\bigcap_{s \in X} s^{-1}x \neq \emptyset$$

which contradicts the weak cancellativity of $\mathcal{S}$.

Inductively, there exists a sequence $(s_n)$ of distinct elements of $\mathcal{S}$ with

$$s_n \in X_x \setminus (\{s_1, \cdots, s_{n-1}\} \cup s_1^{-1}x \cup \cdots \cup s_{n-1}^{-1}x)$$

and

$$s_n^{-1}x \nsubseteq X_x \setminus (\{s_1, \cdots, s_{n-1}\} \cup s_1^{-1}x \cup \cdots \cup s_{n-1}^{-1}x)$$

for all $n \geq 2$. For each $n \geq 1$, choose

$$t_n \in s_n^{-1}x \setminus (\{s_1, \cdots, s_{n-1}\} \cup s_1^{-1}x \cup \cdots \cup s_{n-1}^{-1}x).$$

Then $(t_n)$ is a sequence of distinct elements of $\mathcal{S}$. Define the function $\phi : \mathcal{S} \to \mathbb{C}$ by

$$\phi(s) = \begin{cases} \dfrac{1}{\sqrt{n}} & \text{if } s = s_n \text{ or } s = t_n \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\phi \in \ell^p(\mathcal{S})$; this together with the fact that $x \in \mathcal{S}^2$ yields

$$(\phi * \phi)(x) = \sum_{s,t \in \mathcal{S},\, st=x} \phi(s)\, \phi(t) \geq \sum_{n=1}^{\infty} \phi(s_n)\, \phi(t_n).$$

That is, $(\phi * \phi)(x)$ does not exist, a contradiction. □

Although, a semifinite semigroup is not necessarily even countable, a discrete group $\mathcal{G}$ is semifinite only if $\mathcal{G}$ is finite. Hence, as a result of Theorem 3.2, we have the following corollary.

**Corollary 3.3.** *Let $\mathcal{G}$ be a group and $2 < p < \infty$. Then $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{G})$ if and only if $\mathcal{G}$ is finite.*

At the end, we present some applications of Theorem 3.2.

**Example 3.4.** (a) Let $\mathcal{S}$ be the semigroup of all real numbers with the operation $st = \max\{s, t\}$ for all $s, t \in \mathbb{R}$; hence $s^{-1}t = \{t\}$ for $s \leq t$ and $s^{-1}t = \emptyset$ otherwise. Then $\mathcal{S}$ is not semifinite. Thus for all $p > 2$, there are $\phi, \psi \in \ell^p(\mathcal{S})$ such that $\phi * \psi$ does not to exist.

Now, consider the subsemigroup $\mathcal{T}$ of this semigroup consisting of all natural numbers is semifinite. Thus $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$ and $p > 2$.

(b) Let $\mathcal{S}$ be the semigroup of all natural numbers with general product or plural operation. Then $\mathcal{S}$ is semifinite. Hence $\phi * \psi$ exists for all $\phi, \psi \in \ell^p(\mathcal{S})$ and $p > 2$.

(c) Let $X$ be an arbitrary infinite set and $\mathcal{S}$ be the semigroup of all characteristic functions on $X$ endowed with the pointwise multiplication. Then for each $A, B \subseteq X$ we have $\chi_A^{-1}\chi_B \neq \emptyset$ if and only if $B \subseteq A$. Hence $\mathcal{S}$ is not semifinite. Thus for each $p > 2$, there are $\phi, \psi \in \ell^p(\mathcal{S})$ such that $\phi * \psi$ does not to exist.

## References


1. Abtahi, F., R. Nasr-Isfahani and A. Rejali, On the $L^p$−conjecture for locally compact groups, *Arch. Math.* **89** (2007), 237–242.
2. Crombez, G., *A charactrization of compact groups*, Simon Stevin **53** (1979), 9–12.
3. Crombez, G., An elementary proof about the order of the elements in a discrete group, *Proc. Amer. Math. Soc.* **85** (1983), 59–60.

**4.** Gaudet, R. J., and J. L. Gamlen, An elementary proof of part of a classical conjecture, *Bull. Austral. Math. Soc.* **3** (1970), 285–292.
**5.** Greenleaf, F. P., *Invariant means on locally compact groups and their applications*, Math. Studies **16**, Van Nostrand, New York, 1969.
**6.** Hewitt, E., and K. Ross, *Abstract harmonic analysis I*, Springer-Verlag, New York, 1970.
**7.** Johnson, D. L., A new proof of the $L^p$-conjecture for locally compact groups, *Colloq. Math.* **47** (1982), 101–102.
**8.** Kunze, R., and E. Stein, Uniformly bounded representations and harmonic analysis of the $2 \times 2$ real unimodualr group, *Amer. J. Math.* **82** (1960), 1–62.
**9.** Lohoue, N., Estimations $L^p$ des coefficients de representation et operateurs de convolution, *Adv. Math.* **38** (1980), 178–221.
**10.** Milnes, P., Convolution of $L^p$ functions on non-commutative groups, *Canad. Math. Bull.* **14** (1971), 265–266.
**11.** Rajagopalan, M., On the $l^p$-spaces of a discrete group, *Colloq. Math.* **10** (1963), 49–52.
**12.** Rajagopalan, M., $L_p$-conjecture for locally compact groups I, *Trans. Amer. Math. Soc.* **125** (1966), 216–222.
**13.** Rajagopalan, M., $L_p$-conjecture for locally compact groups II, *Math. Ann.* **169** (1967), 331–339.
**14.** Rajagopalan, M., and W. Zelazko, $L_p$-conjecture for solvable locally compact groups, *J. Indian Math. Soc.* **29** (1965), 87–93.
**15.** Rickert, N. W., Convolution of $L^p$ functions, *Proc. Amer. Math. Soc.* **18** (1967), 762–763.
**16.** Rickert, N. W., Convolution of $L^2$ functions, *Colloq. Math.* **19** (1968), 301–303.
**17.** Saeki, S., The $L^p-$conjecture and Young's inequality, *Illinois. J. Math.* **34** (1990), 615–627.
**18.** Urbanik, K., A proof of a theorem of Zelazko on $L^p$-algebras, *Colloq. Math.* **8** (1961), 121–123.
**19.** Zelazko, W., On the algebras $L^p$ of a locally comapct group, *Colloq. Math.* **8** (1961), 112–120.
**20.** Zelazko, W., A note on $L^p$- algebras, *Colloq. Math.* **10** (1963), 53–56.
**21.** Zelazko, W., On the Burnside problem for locally compact groups, *Symp. Math.* **16** (1975), 409–416.

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