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MONOGENIC FUNCTIONS IN A FINITE-DIMENSIONAL ALGEBRA WITH UNIT AND RADICAL OF MAXIMAL DIMENSIONALITY

S. PLAKSA AND V. SHPAKIVSKYI

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Abstract. We obtain a constructive description of monogenic functions taking values in a finite-dimensional commutative algebra with unit and radical of maximal dimensionality by means of holomorphic functions of the complex variable. We prove that the mentioned monogenic functions have the Gateaux derivatives of all orders, and analogues of classical theorems of the complex analysis hold for them: the Cauchy integral theorem and the Cauchy integral formula, the Taylor expansion and the Morera theorem.

1. Introduction

An effectiveness of the analytic function methods in the complex plane for researching plane potential fields inspires mathematicians to develop analogous methods for spatial fields.

Apparently, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\Delta_3 u(x,y,z) := \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x,y,z)=0 \qquad (1)$$

in that sense that components of hypercomplex functions satisfy the equation (1). He constructed an algebra of noncommutative quaternions over the field of real numbers $\mathbb{R}$, and developing the hypercomplex analysis began.

C. Segre [**1**] constructed an algebra of commutative quaternions over the field $\mathbb{R}$ that can be considered as a two-dimensional commutative algebra over the field of complex numbers $\mathbb{C}$.

A relation between spatial potential fields and analytic functions given in commutative algebras was established by P. W. Ketchum [**2**] who shown that every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3$ satisfies the equation

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(1) in the case where the elements e_1, e_2, e_3 of a commutative algebra satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{2}$$

because

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \equiv \Phi''(\zeta)\;(e_1^2 + e_2^2 + e_3^2) = 0\,, \tag{3}$$

where $\Phi'' := (\Phi')'$ and $\Phi'(\zeta)$ is defined by the equality $d\Phi = \Phi'(\zeta)d\zeta$.

We say that a commutative associative algebra $\mathbb{A}$ is *harmonic* (cf. [**2, 3, 4, 5**]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors $\{e_1, e_2, e_3\}$ satisfying the equality (2) provided that $e_k^2 \neq 0$ for $k = 1, 2, 3$. We say also that such a triad $\{e_1, e_2, e_3\}$ is *harmonic*.

P. W. Ketchum [**2**] considered the C. Segre algebra of quaternions [**1**] as an example of harmonic algebra.

I. P. Mel'nichenko [**3**] noticed that doubly differentiable in the sense of Gateaux functions form the largest algebra of functions Φ satisfying identically the equalities (3), where Φ'' is the Gateaux second derivative of function Φ. He proved that there exist exactly 3 three-dimensional harmonic algebras with unit over the field $\mathbb{C}$ only (see [**3, 4, 5**]).

Constructive descriptions of *monogenic* (i.e. continuous and differentiable in the sense of Gateaux) functions taking values in the mentioned three-dimensional harmonic algebras by means holomorphic functions of the complex variable are obtained in the papers [**6, 7, 8**]. Such descriptions make it possible to prove the infinite differentiability in the sense of Gateaux of monogenic functions and integral theorems for these functions that are analogous to classical theorems of the complex analysis (see, e.g., [**9, 10**]).

Thus, monogenic functions $\Phi(\zeta)$ in every three-dimensional harmonic algebra satisfy the equalities (3) (we shall term such functions by *monogenic potentials*).

Furthermore, in the paper [**6**], we established the following characteristic geometric property of monogenic potentials taking values in the three-dimensional harmonic algebra with two-dimensional radical: every monogenic potential given in a convex domain can be continued to a monogenic potential given in a cylindrical domain.

In this paper we generalize some results of the paper [**6**] to the case of monogenic functions taking values in a commutative finite-dimensional algebra with unit and radical of maximal dimensionality. In particular, we establish the mentioned characteristic geometric property for monogenic potentials belonging to a class more wide than one in the paper [**6**]. As in the papers [**9, 10**], we prove that analogues of classical theorems of the complex analysis hold for monogenic potentials: the Cauchy integral theorem and the Cauchy integral formula, the Taylor expansion and the Morera theorem.

2. MONOGENIC FUNCTIONS IN A FINITE-DIMENSIONAL HARMONIC ALGEBRA

2.1. A finite-dimensional algebra with unit and radical of maximal dimensionality

Let $\mathbb{A}_n$ be a n-dimensional commutative associative Banach algebra over the field $\mathbb{C}$ with the basis $\{1,\rho,\rho^2,\ldots,\rho^{n-1}\}$, where $\rho^n=0$ and $n\geq 3$.

The algebra $\mathbb{A}_n$ have the unique maximal ideal

$$\mathfrak{I}:=\Big\{\sum_{k=1}^{n-1}\lambda_k\rho^k:\lambda_k\in\mathbb{C}\Big\}$$

which is also the radical of $\mathbb{A}_n$. Consider the linear functional $f:\mathbb{A}_n\to\mathbb{C}$ such that the maximal ideal $\mathfrak{I}$ is its kernel and $f(1)=1$.

Consider the following triad in $\mathbb{A}_n$:

$$e_1=1,\qquad e_2=i+\sum_{k=1}^{n_0}\rho^{2k},\qquad e_3=\sum_{k=1}^{n_0}b_{2k-1}\,\rho^{2k-1}\,, \tag{4}$$

where $n_0:=\left[\frac{n-1}{2}\right]$ and the coefficients b_{2k-1} are determined by the following recurrence relations:

$$\begin{aligned} &b_1=1-i\,,\qquad b_3=\frac{1}{4}-\frac{3}{4}\,i\,,\\ &b_{2k-1}=-\frac{1}{2b_1}\Big(k-1+2i+\sum_{j=2}^{k-1}b_{2j-1}b_{2k+1-2j}\Big)\,,\qquad k=3,4,\ldots,n_0\,. \end{aligned} \tag{5}$$

The coefficients $b_1,b_3,\ldots,b_{2n_0-1}$ satisfy the equalities

$$k-1+2i+\sum_{j=1}^{k}b_{2j-1}b_{2k+1-2j}=0\,,\qquad k=1,2,\ldots,n_0\,,$$

that imply the equality (2) for the triad (4). Thus, the triad (4) is harmonic.

Let us note that in the case $n=3$ all harmonic triads in $\mathbb{A}_3$ are described in Theorem 1.6 [**5**].

In what follows, $\zeta:=xe_1+ye_2+ze_3$ and $x,y,z\in\mathbb{R}$.

Let $E_3:=\{\zeta=xe_1+ye_2+ze_3:\ x,y,z\in\mathbb{R}\}$ be the linear span of vectors e_1,e_2,e_3 over the field $\mathbb{R}$.

Inasmuch as the radical $\mathfrak{I}$ contains all noninvertible elements of the algebra $\mathbb{A}_n$, an element $\zeta=xe_1+ye_2+ze_3\in E_3$ is noninvertible in $\mathbb{A}_n$ if and only if $x=y=0$, i.e. (x,y,z) is a point of the axis Oz in the space $\mathbb{R}^3$. Thus, the noninvertible elements in E_3 form the straight line $Z:=\{ze_3:z\in\mathbb{R}\}$.

2.2. Monogenic functions

Some hypercomplex functions given in the algebra $\mathbb{A}_n$ was considered by M. N. Roşculeţ [**11**, p. 85]. We shall consider functions given in domains of E_3.

Let Ω be a domain in $\mathbb{R}^3$. Associate with Ω the domain $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x,y,z) \in \Omega\}$ in E_3.

We say that a continuous function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ is *monogenic* in Ω_ζ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i.e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_n$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_3.$$

$\Phi'(\zeta)$ is the *Gateaux derivative* of the function Φ in the point ζ.

In turn, if Φ' is a monogenic function in the domain Ω_ζ, then we denote the Gateaux derivative of the function Φ' by Φ'' and call Φ'' by the *Gateaux second derivative*. Further, in the same way we define the *Gateaux m-th derivative* $\Phi^{(m)}$.

Consider the decomposition of a function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ with respect to the basis $\{1, \rho, \rho^2, \ldots, \rho^{n-1}\}$:

$$\Phi(\zeta) = \sum_{k=0}^{n-1} W_k(x,y,z)\, \rho^k\,, \tag{6}$$

where by definition $\rho^0 := 1$.

In the case where the functions $W_k : \Omega \to \mathbb{C}$ are $\mathbb{R}$-differentiable in Ω, i.e. for every $(x,y,z) \in \Omega$,

$$W_k(x+\Delta x, y+\Delta y, z+\Delta z) - W_k(x,y,z) = \frac{\partial W_k}{\partial x}\,\Delta x + \frac{\partial W_k}{\partial y}\,\Delta y + \frac{\partial W_k}{\partial z}\,\Delta z +$$

$$+\, o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}\right), \qquad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \to 0\,,$$

the function Φ is monogenic in the domain Ω_ζ if and only if the following Cauchy – Riemann conditions are satisfied in Ω_ζ:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x}\, e_2\,, \quad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\, e_3\,. \tag{7}$$

Below, we shall show that all components W_k of the monogenic function (6) are infinitely $\mathbb{R}$-differentiable in Ω.

2.3. A constructive description of monogenic functions taking values in the algebra $\mathbb{A}_n$

We say that a domain $\Omega_\zeta \subset E_3$ is *convex in the direction of the straight line* Z if the congruent domain $\Omega \subset \mathbb{R}^3$ contains every segment parallel to the axis Oz and connecting two points $(x_1,y_1,z_1), (x_2,y_2,z_2) \in \Omega$.

Let a domain $\Omega_\zeta \subset E_3$ be convex in the direction of the straight line Z.

In what follows, $\xi := f(\zeta) \equiv x + iy$.

A constructive description of monogenic functions taking values in the algebra $\mathbb{A}_3$ by means of holomorphic functions of the complex variable is obtained in the paper [**6**]. More exactly, it is proved in the paper [**6**] that any monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ can be constructed in the form

$$\Phi(\zeta) = F(\xi) + \Big((1-i)zF'(\xi) + F_1(\xi)\Big)\rho_1 + \tag{8}$$

$$+\Big(yF'(\xi) - iz^2F''(\xi) + (1-i)zF_1'(\xi) + F_2(\xi)\Big)\rho_2 \qquad \forall\,\zeta \in \Omega_\zeta$$

by means of three functions F, F_1, F_2 holomorphic in the domain $D := f(\Omega_\zeta)$, where $f(\Omega_\zeta)$ is the image of Ω_ζ under the mapping f.

Below, we obtain a similar constructive description of any monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$.

Let A be the linear operator which assigns a holomorphic function $F : D \to \mathbb{C}$ to every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ by the formula

$$F(\xi) = f(\Phi(\zeta)), \tag{9}$$

where $\xi = f(\zeta) \equiv x + iy$ and $\zeta \in \Omega_\zeta$. The value $F(\xi)$ does not depend on a choice of a point ζ for which $f(\zeta) = \xi$ that can be proved similarly to Lemma 1 [**6**] (see also Lemma 1.1 [**10**] or Lemma 2.1 [**12**]).

Similar operators A which map monogenic functions taking values in certain commutative algebras onto holomorphic functions of the complex variable are explicitly constructed in the papers [**5, 6, 10, 12**]. Furthermore, principal extensions of holomorphic functions of the complex variable are used there as *generalized inverse* operators $A^{(-1)}$ satisfying the equality $AA^{(-1)}A = A$. It was also established for every monogenic function Φ that values of the monogenic function $\Phi - A^{(-1)}A\Phi$ belong to a certain maximal ideal of given algebra. Finally, after describing all monogenic functions taking values in the mentioned ideal, constructive descriptions of monogenic functions by means of holomorphic functions of the complex variable are obtained in the case of certain finite-dimensional algebras (see [**6, 10, 12**]).

To construct explicitly principal extensions of holomorphic functions of the complex variable into the algebra $\mathbb{A}_n$, we use the next auxiliary statements.

Lemma 2.1. *The spectrum of element $\zeta = xe_1 + ye_2 + ze_3 \in E_3 \subset \mathbb{A}_n$ consists of the unique point $t = x + iy$, and the decomposition of resolvent $(t-\zeta)^{-1}$ with respect to the basis $\{1, \rho, \rho^2, \ldots, \rho^{n-1}\}$ is of the form*

$$(t-\zeta)^{-1} = \sum_{k=0}^{n-1} A_k \rho^k \qquad \forall\, t \in \mathbb{C} : \, t \neq x + iy, \tag{10}$$

with the coefficients A_k determined by the following recurrence relations:

$$\begin{aligned} &A_0 = \frac{1}{t-\xi}\,, \qquad A_1 = zb_1(A_0)^2\,, \\ &A_k = A_0\Bigg(y\sum_{j=1}^{k_1} A_{k-2j} + z\sum_{j=0}^{k_0} b_{2j+1}A_{k-2j-1}\Bigg), \quad k = 2, 3, \ldots, n-1, \end{aligned} \tag{11}$$

where $k_1 := [\,\frac{k}{2}\,]$, $k_0 := \left[\frac{k-1}{2}\right]$ and b_{2j+1} are determined by the relations (5).

Proof. Let us determine the coefficients $A_k \in \mathbb{C}$ of decomposition (10). Taking into account the decompositions (4) of elements of harmonic triad $\{e_1, e_2, e_3\}$ with respect to the basis $\{1, \rho, \rho^2, \ldots, \rho^{n-1}\}$, we have

$$1 = (t-\zeta)^{-1}(t-\zeta) = A_0(t-\xi) + \Big(A_1(t-\xi) - zb_1A_0\Big)\rho +$$

$$+\sum_{k=2}^{n-1}\left(A_k(t-\xi)-y\sum_{j=1}^{k_1}A_{k-2j}-z\sum_{j=0}^{k_0}b_{2j+1}\,A_{k-2j-1}\right)\rho^k\,,$$

whence we obtain the following system for finding the coefficients $A_0, A_1, \ldots, A_{n-1}$:

$$A_0(t-\xi)=1, \qquad A_1(t-\xi)-zb_1A_0=0$$

$$A_k(t-\xi)-y\sum_{j=1}^{k_1}A_{k-2j}-z\sum_{j=0}^{k_0}b_{2j+1}A_{k-2j-1}=0\,. \tag{12}$$

The system (12) is solvable if $t-\xi\neq 0$, i.e. the spectrum of element ζ consists of the unique point $t=\xi$, and the relations (11) follow from the equalities (12). □

Lemma 2.2. *The coefficients* (11) *are represented in the form*

$$A_k=\frac{1}{(t-\xi)^{k+1}}\sum_{m=0}^{k}(t-\xi)^m\,P_{k,m}(y,z)\,, \qquad k=0,1,\ldots,n-1, \tag{13}$$

where $P_{k,m}$ is a homogeneous polynomial of the $(k-m)$-th power. Moreover, $P_{k,m}$ can be determined by the following relations:

$$P_{0,0}(y,z)\equiv 1, \quad P_{k,0}(y,z)=(zb_1)^k, \quad P_{k,k}(y,z)\equiv 0 \quad \textit{for } k=1,2,\ldots,n-1\,,$$

$$\begin{aligned}P_{k,m}(y,z)&=y\sum_{r=0}^{k-m-1}(zb_1)^r\sum_{p=1}^{k}P_{k-2p-r,m-2p+1}(y,z)+\\&+z\sum_{r=0}^{k-m-1}(zb_1)^r\sum_{p=1}^{k}b_{2p+1}P_{k-2p-r-1,m-2p}(y,z) \quad \textit{for } k>m>0\,,\end{aligned} \tag{14}$$

where $P_{q,j}(y,z)\equiv 0$ for all q and $j<0$.

Proof. Let us prove the representation (13) by the mathematical induction. It is clear that A_0 is represented in the form (13) with $P_{0,0}(y,z)\equiv 1$. Further, supposing that all $A_0, A_1, \ldots, A_{k-1}$ are represented in the form (13), we shall prove that A_k is also represented in the form (13). Substituting the expressions (13) of $A_0, A_1, \ldots, A_{k-1}$ into the equality (11), we obtain

$$\begin{aligned}A_k&=\frac{1}{t-\xi}\Bigg(y\sum_{j=1}^{k_1}\frac{1}{(t-\xi)^{k-2j+1}}\sum_{r=0}^{k-2j}(t-\xi)^rP_{k-2j,r}(y,z)+\\&+z\sum_{j=0}^{k_0}b_{2j+1}\frac{1}{(t-\xi)^{k-2j}}\sum_{r=0}^{k-2j-1}(t-\xi)^rP_{k-2j-1,r}(y,z)\Bigg)=\\&=\frac{1}{(t-\xi)^{k+1}}\Bigg(\sum_{j=1}^{k_1}\sum_{r=0}^{k-2j}(t-\xi)^{r+2j-1}\,yP_{k-2j,r}(y,z)+\\&+\sum_{j=0}^{k_0}\sum_{r=0}^{k-2j-1}(t-\xi)^{r+2j}\,zb_{2j+1}P_{k-2j-1,r}(y,z)\Bigg).\end{aligned}$$

The last expression can be represented as (13) with $P_{k,k}(y,z)\equiv 0$ for $k=1,2,\ldots,n-1$ and

$$P_{k,m}(y,z)=y\sum_{p=1}^{k}P_{k-2p,m-2p+1}(y,z)+z\sum_{p=1}^{k}b_{2p+1}P_{k-2p-1,m-2p}(y,z)+ \tag{15}$$
$$+zb_1P_{k-1,m}(y,z)\,,\qquad k>m\geq 0,$$

where we set by definition $P_{q,j}(y,z)\equiv 0$ for all q and $j<0$. Thus, $P_{k,m}$ is a homogeneous polynomial of the $(k-m)$-th power, and the representation (13) is proved.

It follows from (15) that $P_{k,0}(y,z)=zb_1P_{k-1,0}(y,z)$ for $k>0$ and, consequently, $P_{k,0}(y,z)=(zb_1)^k$ for $k=1,2,\ldots,n-1$.

In the case $k-1=m>0$ the equality (15) coincides with (14) because $P_{m,m}(y,z)\equiv 0$.

To obtain the equality (14) in the case $k-1>m>0$, primarily, we substitute an expression $P_{k-1,m}$ of the form (15) into the equality (15) and get

$$P_{k,m}(y,z)=y\sum_{p=1}^{k}P_{k-2p,m-2p+1}(y,z)+z\sum_{p=1}^{k}b_{2p+1}P_{k-2p-1,m-2p}(y,z)+$$
$$+zb_1\left(y\sum_{p=1}^{k}P_{k-2p-1,m-2p+1}(y,z)+z\sum_{p=1}^{k}b_{2p+1}P_{k-2p-2,m-2p}(y,z)\right)+$$
$$+(zb_1)^2P_{k-2,m}(y,z)\,.$$

Now, the last equality coincides with (14) in the case $k-2=m>0$.

Next, in the case $k-2>m>0$ we substitute an expression $P_{k-2,m}$ of the form (15) into the last equality and get

$$P_{k,m}(y,z)=y\sum_{r=0}^{2}(zb_1)^r\sum_{p=1}^{k}P_{k-2p-r,m-2p+1}(y,z)+$$
$$+z\sum_{r=0}^{2}(zb_1)^r\sum_{p=1}^{k}b_{2p+1}P_{k-2p-r-1,m-2p}(y,z)+(zb_1)^3P_{k-3,m}(y,z)\,.$$

Finally, continuing similar operations, after $k-m-1$ steps we get the equality (14). □

For instance, we adduce the polynomials $P_{k,m}$ for $k=m+1,m+2,\ldots,n-1$ and $m=1,2,3$:

$$P_{k,1}(y,z)=(k-1)\,y(zb_1)^{k-2},$$
$$P_{k,2}(y,z)=\frac{1}{2}\,(k-2)(k-3)\,y^2(zb_1)^{k-4}+(k-2)\,zb_3(zb_1)^{k-3},$$
$$P_{k,3}(y,z)=(k-3)\,y(zb_1)^{k-4}+(k-3)(k-4)\,yzb_3(zb_1)^{k-5}+$$
$$+\frac{1}{6}\,(k-3)(k-4)(k-5)\,y^3(zb_1)^{k-6}\,.$$

The following equality is an evident consequence of the equalities (10), (13):

$$(t-\zeta)^{-1}=\sum_{k=0}^{n-1}\rho^k\sum_{m=0}^{k}(t-\xi)^{m-k-1}P_{k,m}(y,z)\qquad\forall\, t\in\mathbb{C}:\, t\neq x+iy\,. \tag{16}$$

In the following theorem we construct the principal extension of holomorphic function $F:D\to\mathbb{C}$ into the cylindrical domain $\Pi_\zeta:=\{\zeta\in E_3: f(\zeta)\in D\}$ in an explicit form.

Theorem 2.3. *Let a function $F:D\to\mathbb{C}$ be holomorphic in a domain $D\subset\mathbb{C}$. Then the principal extension of F into the domain Π_ζ can be explicitly constructed in the form*

$$\frac{1}{2\pi i}\int\limits_{\Gamma}F(t)(t-\zeta)^{-1}\,dt=\sum_{k=0}^{n-1}\rho^k\sum_{m=0}^{k}P_{k,m}(y,z)\,\frac{F^{(k-m)}(\xi)}{(k-m)!}\,, \tag{17}$$

where Γ is an arbitrary closed Jordan rectifiable curve in D that is homotopic to the point ξ and embraces this point, and $P_{k,m}$ is the same as in Lemma 2.2.

Proof. Using the equality (16), we obtain

$$\frac{1}{2\pi i}\int\limits_{\Gamma}F(t)(t-\zeta)^{-1}\,dt=\sum_{k=0}^{n-1}\rho^k\sum_{m=0}^{k}\frac{P_{k,m}(y,z)}{2\pi i}\int\limits_{\Gamma}\frac{F(t)}{(t-\xi)^{k-m+1}}\,dt=$$

$$=\sum_{k=0}^{n-1}\rho^k\sum_{m=0}^{k}P_{k,m}(y,z)\,\frac{F^{(k-m)}(\xi)}{(k-m)!}\,.$$

□

In the following theorem we describe all monogenic functions $\Phi:\Omega_\zeta\to\mathbb{A}_n$ by means of principal extensions of holomorphic functions of the complex variable.

Theorem 2.4. *If a domain $\Omega_\zeta\subset E_3$ is convex in the direction of the straight line Z, then any monogenic function $\Phi:\Omega_\zeta\to\mathbb{A}_n$ can be expressed in the form*

$$\Phi(\zeta)=\sum_{k=0}^{n-1}\rho^k\,\frac{1}{2\pi i}\int\limits_{\Gamma}F_k(t)(t-\zeta)^{-1}\,dt\qquad\forall\zeta\in\Omega_\zeta\,, \tag{18}$$

where $F_k:D\to\mathbb{C}$ is a holomorphic function in the domain $D:=f(\Omega_\zeta)$ and the curve Γ is the same as in Theorem 2.3.

Proof. We set $F_0:=A\Phi$, i.e. the function F_0 is defined by the formula (9), where $F=F_0$. It is easy to see that the monogenic function

$$\Phi_0(\zeta):=\Phi(\zeta)-\frac{1}{2\pi i}\int\limits_{\Gamma}F_0(t)(t-\zeta)^{-1}\,dt$$

belongs to the kernel of the operator A, i.e. $\Phi_0(\zeta)\in\mathfrak{I}$ for all $\zeta\in\Omega_\zeta$.

Therefore, the function Φ_0 is of the form

$$\Phi_0(\zeta)=\sum_{k=1}^{n-1} V_k(x,y,z)\,\rho^k\,, \tag{19}$$

where $V_k:\Omega\to\mathbb{C}$, and the Cauchy – Riemann conditions (7) are satisfied with $\Phi=\Phi_0$.

Substituting the expressions (4), (19) into the equalities (7) and taking into account the uniqueness of decomposition of element of $\mathbb{A}_n$ with respect to the basis $\{1,\rho,\rho^2,\ldots,\rho^{n-1}\}$, we get the following system for the functions $V_1,V_2,\ldots,V_{n-1}$:

$$\begin{aligned}
&\frac{\partial V_1}{\partial y}=i\,\frac{\partial V_1}{\partial x}\,,\\
&\frac{\partial V_2}{\partial y}=i\,\frac{\partial V_2}{\partial x}\,,\\
&\frac{\partial V_k}{\partial y}=i\,\frac{\partial V_k}{\partial x}+\sum_{j=1}^{k_0}\frac{\partial V_{k-2j}}{\partial x}\,,\qquad k_0:=\left[\tfrac{k-1}{2}\right]\,,\quad k=3,4,\ldots,n-1\,,\\
&\frac{\partial V_1}{\partial z}=0\,,\\
&\frac{\partial V_k}{\partial z}=\sum_{j=1}^{k_1} b_{2j-1}\,\frac{\partial V_{k-2j+1}}{\partial x}\,,\qquad k_1:=\left[\,\tfrac{k}{2}\,\right],\qquad k=2,3,\ldots,n-1\,.
\end{aligned} \tag{20}$$

Taking into account Theorem 6 [**13**], we get from the first and the fourth equations of the system (20) that $V_1(x,y,z)\equiv F_1(\xi)$, where F_1 is a holomorphic function in the domain D.

Now, consider the function

$$\Phi_1(\zeta):=\Phi(\zeta)-\frac{1}{2\pi i}\int_\Gamma F_0(t)(t-\zeta)^{-1}\,dt-\rho\,\frac{1}{2\pi i}\int_\Gamma F_1(t)(t-\zeta)^{-1}\,dt$$

which can be represented in the form

$$\Phi_1(\zeta)=\sum_{k=2}^{n-1}\widetilde{V}_k(x,y,z)\,\rho^k,$$

where $\widetilde{V}_k:\Omega\to\mathbb{C}$.

Inasmuch as Φ_1 is a monogenic function in Ω_ζ, the functions $\widetilde{V}_2,\widetilde{V}_3,\ldots,\widetilde{V}_{n-1}$ satisfy the system (20), where $V_1\equiv 0$, $V_k=\widetilde{V}_k$ for $k=2,3,\ldots,n-1$. Therefore, similarly to the function $V_1(x,y,z)\equiv F_1(\xi)$, the function $\widetilde{V}_2$ satisfies the equations

$$\frac{\partial \widetilde{V}_2}{\partial y}=i\,\frac{\partial \widetilde{V}_2}{\partial x}\,,\qquad \frac{\partial \widetilde{V}_2}{\partial z}=0$$

and is of the form $\widetilde{V}_2(x,y,z)\equiv F_2(\xi)$, where F_2 is a holomorphic function in the domain D.

In such a way, step by step, considering the functions

$$\Phi_j(\zeta) := \Phi(\zeta) - \sum_{k=0}^{j} \rho^k \frac{1}{2\pi i} \int_{\Gamma} F_k(t)(t-\zeta)^{-1}\, dt$$

for $j = 2, 3, \ldots, n-2$, we get the representation (18) of the function Φ. □

It is evident that the following statement follows from the equalities (18), (17).

Theorem 2.5. *If a domain $\Omega_\zeta \subset E_3$ is convex in the direction of the straight line Z, then every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ can be continued to a function monogenic in the domain Π_ζ.*

The following statement is true for monogenic functions in an arbitrary domain Ω_ζ.

Theorem 2.6. *For every monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ in an arbitrary domain $\Omega_\zeta \subset E_3$, the Gateaux m-th derivatives $\Phi^{(m)}$ are monogenic functions in Ω_ζ for all m.*

Proof. Consider an arbitrary point $\zeta_0 \in \Omega_\zeta$ and a ball $\mho_\zeta \subset \Omega_\zeta$ with the center in the point ζ_0. Inasmuch as $\mho_\zeta$ is a convex set, we have the equality (18) in $\mho_\zeta$, where the integrals have the Gateaux m-th derivatives for all m that are continuous functions in $\mho_\zeta$. Thus, the Gateaux m-th derivative $\Phi^{(m)}$ is a monogenic function in $\mho_\zeta$ for any m. □

It follows from Theorem 2.6 that every monogenic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3 \in \Omega_\zeta$ satisfies the three-dimensional Laplace equation due to the equalities (2) and (3), i.e. $\Phi(\zeta)$ is a monogenic potential. In this case, all components W_k of the decomposition (6) are infinitely $\mathbb{R}$-differentiable functions in Ω, and, moreover, the real and imaginary parts of the functions $W_0, W_1, \ldots, W_{n-1}$ form $2n$-tuple of three-dimensional harmonic functions in the domain Ω.

Using the integral expression (18) of monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ in the case where a domain $\Omega_\zeta \subset E_3$ is convex in the direction of the straight line Z, we obtain the following expression for the Gateaux m-th derivative $\Phi^{(m)}$:

$$\Phi^{(m)}(\zeta) = \sum_{k=0}^{n-1} \rho^k \frac{m!}{2\pi i} \int_{\Gamma} F_k(t)\Big((t-\zeta)^{-1}\Big)^{m+1} dt \qquad \forall \zeta \in \Omega_\zeta\,.$$

It follows from Theorem 2.5 that monogenic potentials $\Phi : \Omega_\zeta \to \mathbb{A}_n$ have the following characteristic geometric property: being given in a convex domain Ω_ζ, they can be continued to monogenic potentials given in the cylindrical domain Π_ζ.

Combining the expressions (18) and (17), one can obtain a constructive description of any monogenic potential $\Phi : \Omega_\zeta \to \mathbb{A}_n$ by means of n holomorphic functions that is similar to the constructive description (8) of monogenic potentials $\Phi : \Omega_\zeta \to \mathbb{A}_3$. Comparing the form of expressions (18), (17) with the form of expression (8), it is easy to see that monogenic potentials $\Phi : \Omega_\zeta \to \mathbb{A}_n$ form a class more wide than the class of monogenic potentials $\Phi : \Omega_\zeta \to \mathbb{A}_3$.

2.4. Integral theorems for monogenic functions

In the paper [**14**], for functions differentiable in the sense of Lorch in an arbitrary convex domain of a commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the Cauchy integral theorem and the Cauchy integral formula, the Taylor expansion and the Morera theorem) are established. The convexity of the domain in the mentioned results from [**14**] is withdrawn by E. K. Blum [**15**].

In this paper we establish similar results for monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_n$ given only in a domain Ω_ζ of the linear span E_3 instead of domain of whole algebra.

Let us note that a priori the differentiability of the function Φ in the sense of Gateaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Moreover, note that the Cauchy integral formula established in the papers [**14, 15**] is not applicable to a monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ because it deals with an integration along a curve on which the function Φ is not given, generally speaking.

Let γ be a Jordan rectifiable curve in $\mathbb{R}^3$. We say that the congruent curve $\gamma_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \gamma\}$ is a Jordan rectifiable curve in E_3.

For a continuous function $\Phi : \gamma_\zeta \to \mathbb{A}_n$ of the form (6), where $(x, y, z) \in \gamma$ and $W_k(x, y, z) = U_k(x, y, z) + iV_k(x, y, z)$ with real-valued functions U_k, V_k, we define an integral along the curve γ_ζ by the equality

$$\int\limits_{\gamma_\zeta} \Phi(\zeta)\, d\zeta := \sum_{k=0}^{n-1} \rho^k \int\limits_\gamma U_k(x, y, z)\, dx + \sum_{k=0}^{n-1} e_2\, \rho^k \int\limits_\gamma U_k(x, y, z)\, dy +$$

$$+ \sum_{k=0}^{n-1} e_3\, \rho^k \int\limits_\gamma U_k(x, y, z)\, dz + i \sum_{k=0}^{n-1} \rho^k \int\limits_\gamma V_k(x, y, z)\, dx +$$

$$+ i \sum_{k=0}^{n-1} e_2\, \rho^k \int\limits_\gamma V_k(x, y, z)\, dy + i \sum_{k=0}^{n-1} e_3\, \rho^k \int\limits_\gamma V_k(x, y, z)\, dz\,,$$

where $d\zeta := e_1 dx + e_2 dy + e_3 dz$.

We understand a triangle $\triangle$ in $\mathbb{R}^3$ as a plane figure bounded by three line segments connecting three its vertices. Denote by $\partial\triangle$ the boundary of triangle $\triangle$ in relative topology of its plane.

Let Ω_ζ be a domain in E_3 and $\Phi : \Omega_\zeta \to \mathbb{A}_n$ be a monogenic function in Ω_ζ. For every triangle $\triangle \subset \Omega$, using the classic Stokes formula and the equalities (7), we obtain immediately the following equality:

$$\int\limits_{\partial\triangle_\zeta} \Phi(\zeta)\, d\zeta = 0\,, \tag{21}$$

where $\partial\triangle_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \partial\triangle\}$.

Now, we can prove the following theorem similarly to the proof of Theorem 3.2 [**15**].

Theorem 2.7. *Let $\Phi : \Omega_\zeta \to \mathbb{A}_n$ be a monogenic function in a domain $\Omega_\zeta \subset E_3$. Then for every closed Jordan rectifiable curve γ_ζ homotopic to a point in Ω_ζ, the following equality holds:*

$$\int_{\gamma_\zeta} \Phi(\zeta)\, d\zeta = 0\,.$$

For functions taking values in the algebra $\mathbb{A}_n$, the following Morera theorem can be established in the usual way.

Theorem 2.8. *If a function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ is continuous in a domain $\Omega_\zeta \subset E_3$ and satisfies the equality* (21) *for every triangle $\triangle \subset \Omega$, then the function Φ is monogenic in the domain Ω_ζ.*

Now, consider a domain Ω_ζ which is convex in the direction of the straight line Z. Let $\zeta_0 := x_0e_1 + y_0e_2 + z_0e_3$ and $(x_0, y_0, z_0) \in \Omega$. In a neighborhood of (x_0, y_0, z_0) contained in Ω, let us take a circle C with the center at the point (x_0, y_0, z_0). We assume that the circle C embraces the straight line $\{(x_0, y_0, z) : z \in \mathbb{R}\}$.

We say that a curve $\gamma_\zeta \subset \Omega_\zeta$ *embraces once the straight line* $\{\zeta_0 + ze_3 : z \in \mathbb{R}\}$ if the congruent curve γ is homotopic to the circle C in the domain $\Omega \setminus \{(x_0, y_0, z) : z \in \mathbb{R}\}$.

The following theorem can be proved in such a way as Theorem 5 [**9**] (see also Theorem 1.14 [**10**]).

Theorem 2.9. *Let a domain Ω_ζ be convex in the direction of the straight line Z and $\Phi : \Omega_\zeta \to \mathbb{A}_n$ be a monogenic function in the domain Ω_ζ. Then for every point $\zeta_0 \in \Omega_\zeta$ the following equality is true:*

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \int_{\gamma_\zeta} \Phi(\zeta)\, (\zeta - \zeta_0)^{-1}\, d\zeta, \tag{22}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ that embraces once the straight line $\{\zeta_0 + ze_3 : z \in \mathbb{R}\}$.

Using the formula (22), in the usual way, we obtain the Taylor expansion

$$\Phi(\zeta) = \sum_{k=0}^{\infty} c_k\, (\zeta - \zeta_0)^k, \qquad c_k \in \mathbb{A}_n, \tag{23}$$

of any monogenic function $\Phi : \Omega_\zeta \to \mathbb{A}_n$ in a certain neighborhood of every point ζ_0 belonging to an arbitrary domain $\Omega_\zeta \subset E_3$.

Thus, the following theorem giving different equivalent definitions of monogenic functions $\Phi : \Omega_\zeta \to \mathbb{A}_n$ is true:

Theorem 2.10. *A function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ is a monogenic in an arbitrary domain $\Omega_\zeta \subset E_3$ if and only if one of the following conditions is satisfied:*

(I) *the components $W_1, W_2, \ldots, W_{n-1}$ of the decomposition* (6) *of the function Φ are $\mathbb{R}$-differentiable in Ω and the conditions* (7) *are satisfied in the domain Ω_ζ;*

(II) *the function* Φ *is continuous in* Ω_ζ *and satisfies the equality* (21) *for every triangle* $\triangle \subset \Omega$*;*

(III) *for every* $\zeta_0 \in \Omega_\zeta$ *there exists a neighborhood, in which the function* Φ *is expressed as the sum of the power series* (23)*;*

(IV) *in every ball* $\mho_\zeta \subset \Omega_\zeta$ *the function* Φ *is expressed in the form* (18)*, where* $F_0, F_1, \ldots, F_{n-1}$ *are some holomorphic functions in the domain* $D := f(\mho_\zeta)$ *and the curve* Γ *is the same as in Theorem 2.3.*

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S. Plaksa, Institute of Mathematics of the National Academy of Sciences of Ukraine, Tereshchenkivska Street 3, Kiev, Ukraine, *e-mail*: `plaksa@imath.kiev.ua`,
`http://www.imath.kiev.ua/~complex/sotrudniki/plaksa/Plaksa_en.html`

V. Shpakivskyi, Institute of Mathematics of the National Academy of Sciences of Ukraine, Tereshchenkivska Street 3, Kiev, Ukraine, *e-mail*: `shpakivskyi@mail.ru`,
`http://www.imath.kiev.ua/~complex/sotrudniki/Shpakivskyi/Shpakivskyi_en.html`