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SUMS OF PRODUCTS OF CAUCHY POLYNOMIALS, INCLUDING POLY-CAUCHY POLYNOMIALS

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Abstract. We investigate sums of products of Cauchy polynomials including poly-Cauchy polynomials. Recently, the concept of poly-Cauchy polynomials has been given by the author as a generalization of the classical Cauchy number and polynomial. A relation among these sums and explicit expressions of sums of two and three products are also given.

1. Introduction

The Bernoulli polynomials $B_n(z)$ are defined by the generating function

$$\frac{xe^{xz}}{e^x-1}=\sum_{n=0}^\infty B_n(z)\frac{x^n}{n!}\,.$$

These polynomials satisfy the identity (see e.g. [**3**], [**5**, Ch. 50], [**14**])

$$(1)\quad \sum_{l=0}^n\binom{n}{l}B_l(x)B_{n-l}(y)=n(x+y-1)B_{n-1}(x+y)-(n-1)B_n(x+y)\quad (n\ge 1)\,.$$

As an analogous result in Bernoulli polynomials, we shall show the following result in Cauchy polynomials (of the first kind) (see e.g. [**1**]).

Theorem 1.1.

$$\sum_{l=0}^n\binom{n}{l}c_l(x)c_{n-l}(y)=-n(x+y+n-2)c_{n-1}(x+y)-(n-1)c_n(x+y)\quad (n\ge 1)\,,$$

where the Cauchy polynomials $c_n(z)$ are defined by the generating function

$$\frac{1}{(1+x)^z}\frac{x}{\ln(1+x)}=\sum_{n=0}^\infty c_n(z)\frac{x^n}{n!}\,.$$

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Remark. *If $x = y = 0$, then the identity in the above theorem is reduced to*

$$\sum_{l=0}^{n} \binom{n}{l} c_l c_{n-l} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \ge 1)\,,$$

which is obtained in [**15**, Theorem 2.1].

For this purpose, we consider the function $T_m^{(k)}(n; z_1, \dots, z_m)$ for $m \ge 1$, $k \in \mathbb{Z}$, $n \ge 0$

$$T_m^{(k)}(n; z_1, \dots, z_m) := \sum_{\substack{i_1+\cdots+i_m=n\\ i_1,\dots,i_m \ge 0}} \binom{n}{i_1, \dots, i_m} \underbrace{c_{i_1}(z_1) \dots c_{i_{m-1}}(z_{m-1})}_{m-1} c_{i_m}^{(k)}(z_m)\,,$$

where $\binom{n}{i_1,\dots,i_m}$ is the multinomial coefficient defined by

$$\binom{n}{i_1, \dots, i_m} = \frac{n!}{i_1! \dots i_m}\,.$$

Here, $c_n^{(k)}(z)$ are the *poly-Cauchy polynomials* (of the first kind) ($n \ge 0$, $k \in \mathbb{Z}$) defined by the generating function

$$\frac{\mathrm{Lif}_k(\ln(1+x))}{(1+x)^z} = \sum_{n=0}^{\infty} c_n^{(k)}(z) \frac{x^n}{n!}\,,$$

where

$$\mathrm{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}$$

is the k-th *polylogarithm factorial* function introduced in [**10, 7**]. When $k = 1$, $c_n^{(1)}(z) = c_n(z)$ are the Cauchy polynomials of the first kind. When $z = 0$, $c_n^{(k)}(0) = c_n^{(k)}$ are the poly-Cauchy numbers of the first kind, which are also introduced in [**10**]. Notice that poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ can be expressed explicitly in terms of the (unsigned) Stirling numbers of the first kind $\left[{n \atop m}\right]$:

$$c_n^{(k)}(z) = \sum_{m=0}^{n} \left[{n \atop m}\right] (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{(m-i+1)^k}\,.$$

([**7**, Theorem 1]).

Then we shall show the following.

Theorem 1.2. *For any integer k, we have for $n = m, m+1, \dots$*

$$\sum_{l=0}^{m} (-1)^{m-l} \left[{m+1 \atop l+1}\right] T_{m+1}^{(k-l)}(n; z_1, \dots, z_{m+1})$$
$$= \sum_{r=0}^{m} \sum_{i=0}^{n-m} \frac{n!}{i!} \left\{{m \atop r}\right\} \sum_{j=0}^{r} \binom{r}{j} \binom{j}{n-m-i} \underbrace{z(z-1)\dots(z-r+j+1)}_{r-j} c_{j+i}^{(k)}(z)\,,$$

where $z = z_1 + \cdots + z_{m+1}$. *For* $0 \le n \le m-1$ *we have*

$$\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1 \\ l+1 \end{bmatrix} T_{m+1}^{(k-l)}(n;z_1,\dots,z_{m+1}) = 0\,.$$

Remark. *If* $z_1 = \cdots = z_{m+1} = 0$, *then we have*

$$\begin{aligned}&\sum_{l=0}^{m}(-1)^{m-l}\begin{bmatrix} m+1 \\ l+1 \end{bmatrix} T_{m+1}^{(k-l)}(n)\\ &= \sum_{r=0}^{m}\sum_{i=0}^{n-m}\frac{n!}{i!}\left\{ m \atop r \right\}\binom{r}{r}\binom{r}{n-m-i}c_{r+i}^{(k)}\\ &= \sum_{r=0}^{m}\sum_{i=0}^{n-m}\frac{n!}{i!}\binom{r}{n-m-i}\left\{ m \atop r \right\}c_{r+i}^{(k)}\,,\end{aligned}$$

which is [**12**, Theorem 1].

The generating function of $T_m^{(k)}(n;z_1,\dots,z_m)$ is given by

$$\frac{1}{(1+x)^{z_1+\cdots+z_{m-1}+z_m}}\left(\frac{x}{\ln(1+x)}\right)^{m-1}\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty}T_m^{(k)}(n;z_1,\dots,z_m)\frac{x^n}{n!}\,.$$

Put

$$G_k(x) := \mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty}c_n^{(k)}\frac{x^n}{n!}\,.$$

Since

$$\mathrm{Lif}_1(t) = \frac{e^t-1}{t},\quad \mathrm{Lif}_0(t) = e^t\quad\text{and}\quad \mathrm{Lif}_{-1}(t) = (t+1)e^t\,,$$

we have

$$G_1(x) = \frac{x}{\ln(1+x)},\quad G_0(x) = 1+x\quad\text{and}\quad G_{-1}(x) = (1+x)(\ln(1+x)+1)\,.$$

Since

$$x^m\frac{d^l}{dx^l}G_k(x) = \sum_{i=0}^{\infty}c_{l+i}^{(k)}\frac{x^{m+i}}{i!}\quad(m,l\ge 0,\ k\in\mathbb{Z})\,, \tag{2}$$

the coefficient of x^n in

$$x^m\frac{d^l}{dx^l}G_k(x)$$

is equal to

$$\begin{cases}\dfrac{c_{n-m+l}^{(k)}}{(n-m)!} & (n\ge m);\\ 0 & (0\le n\le m-1)\,.\end{cases}$$

We need the following Lemma ([**12**, Lemma 2]) in order to prove Theorem 1.2.

Lemma 1.3. *For an integer k and a positive integer m, we have*

$$\left(\left\{{m \atop m}\right\}\frac{d^m}{dx^m}+\left\{{m \atop m-1}\right\}\frac{1}{1+x}\frac{d^{m-1}}{dx^{m-1}}+\dots+\left\{{m \atop 1}\right\}\frac{1}{(1+x)^{m-1}}\frac{d}{dx}\right)G_k(x)$$
$$=\frac{1}{(1+x)^m(\ln(1+x))^m}\sum_{l=0}^m(-1)^{m-l}\left[{m+1 \atop l+1}\right]G_{k-l}(x)\,.$$

Lemma 1.4.

$$\frac{1}{(1+x)^z}\sum_{n=0}^\infty c_{n+r}^{(k)}\frac{x^n}{n!}=\sum_{j=0}^r\binom{r}{j}\frac{\overbrace{z(z-1)\dots(z-r-j+1)}^{r-j}}{(1+x)^{r-j}}\sum_{n=0}^\infty c_{n+j}^{(k)}(z)\frac{x^n}{n!}\,.$$

Proof. Notice that

$$\frac{1}{(1+x)^z}\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}=\sum_{n=0}^\infty c_n^{(k)}(z)\frac{x^n}{n!}\,.$$

By induction, we can prove that

$$\frac{1}{(1+x)^z}\frac{d^r}{dx^r}\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}=\sum_{i=0}^r\binom{r}{i}\frac{\overbrace{z(z-1)\dots(z-i+1)}^{i}}{(1+x)^{i}}\frac{d^{r-i}}{dx^{r-i}}\sum_{n=0}^\infty c_n^{(k)}(z)\frac{x^n}{n!}\,.$$

Since

$$\frac{d^r}{dx^r}\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}=\sum_{n=0}^\infty c_{n+r}^{(k)}\frac{x^n}{n!}\,,$$

we get

$$\frac{1}{(1+x)^z}\sum_{n=0}^\infty c_{n+r}^{(k)}\frac{x^n}{n!}=\sum_{j=0}^r\binom{r}{j}\frac{\overbrace{z(z-1)\dots(z-j+1)}^{j}}{(1+x)^{j}}\sum_{n=0}^\infty c_{n+r-j}^{(k)}(z)\frac{x^n}{n!}$$
$$=\sum_{j=0}^r\binom{r}{j}\frac{\overbrace{z(z-1)\dots(z-r+j+1)}^{r-j}}{(1+x)^{r-j}}\sum_{n=0}^\infty c_{n+j}^{(k)}(z)\frac{x^n}{n!}\,.$$

□

Proof of Theorem 1.2. By Lemma 1.3 and Lemma 1.4, we obtain

$$\begin{aligned}
&\sum_{n=0}^\infty\sum_{l=0}^m(-1)^{m-l}\begin{bmatrix}m+1\\l+1\end{bmatrix}T_{m+1}^{(k-l)}(n;z_1,\dots,z_{m+1})\frac{x^n}{n!}\\
&=\frac{1}{(1+x)^z}\left(\frac{x}{\ln(1+x)}\right)^m\sum_{l=0}^m(-1)^{m-l}\begin{bmatrix}m+1\\l+1\end{bmatrix}G_{k-l}(x)\quad(z=z_1+\cdots+z_{m+1})\\
&=\frac{x^m}{(1+x)^z}\left((1+x)^m\left\{m\atop m\right\}\frac{d^m}{dx^m}+(1+x)^{m-1}\left\{m\atop m-1\right\}\frac{d^{m-1}}{dx^{m-1}}\right.\\
&\qquad\qquad\left.+\cdots+(1+x)\left\{m\atop 1\right\}\frac{d}{dx}\right)G_k(x)\\
&=\frac{x^m}{(1+x)^z}\sum_{r=0}^m(1+x)^r\left\{m\atop r\right\}\sum_{i=0}^\infty c_{r+i}^{(k)}\frac{x^i}{i!}\\
&=x^m\sum_{r=0}^m(1+x)^r\left\{m\atop r\right\}\sum_{j=0}^r\binom{r}{j}\frac{z(z-1)\dots(z-r+j+1)}{(1+x)^{r-j}}\sum_{\nu=0}^\infty c_{\nu+j}^{(k)}(z)\frac{x^\nu}{\nu!}\\
&=x^m\sum_{r=0}^m\left\{m\atop r\right\}\sum_{j=0}^r\binom{r}{j}z(z-1)\dots(z-r+j+1)\cdot(1+x)^j\sum_{\nu=0}^\infty c_{\nu+j}^{(k)}(z)\frac{x^\nu}{\nu!}\\
&=x^m\sum_{r=0}^m\left\{m\atop r\right\}\sum_{j=0}^r\binom{r}{j}z(z-1)\dots(z-r+j+1)\sum_{n=0}^\infty\sum_{i=0}^n\binom{j}{n-i}c_{j+i}^{(k)}(z)\frac{x^n}{i!}\\
&=\sum_{n=m}^\infty\sum_{r=0}^m\sum_{i=0}^{n-m}\frac{n!}{i!}\left\{m\atop r\right\}\\
&\qquad\sum_{j=0}^r\binom{r}{j}\binom{j}{n-m-i}\underbrace{z(z-1)\dots(z-r+j+1)}_{r-j}c_{j+i}^{(k)}(z)\frac{x^n}{n!}\,.
\end{aligned}$$

□

If we put $m=1$ in Theorem 1.2, we have the following.

Corollary 1.5. *For $n\ge 0$ and $k\in\mathbb{Z}$*

$$\sum_{i=0}^n\binom{n}{i}c_i(x)\bigl(c_{n-i}^{(k-1)}(y)-c_{n-i}^{(k)}(y)\bigr)=n\bigl((x+y+n-1)c_{n-1}^{(k)}(x+y)+c_n^{(k)}(x+y)\bigr)\,.$$

2. Explicit formulae for $T_2^{(k)}(n;x,y)$

Theorem 2.1. *For $n\ge 1$ we have*

$$T_2^{(0)}(n;x,y)=c_n^{(1)}(x+y-1)=\sum_{m=0}^n\left[{n\atop m}\right](-1)^{n-m}\sum_{i=0}^m\binom{m}{i}\frac{(-x-y+1)^i}{m-i+1}\,, \tag{3}$$

$$\begin{aligned}T_2^{(k)}(n;x,y)&=c_n^{(1)}(x+y-1)\\&\quad-n\sum_{j=1}^k\bigl(c_n^{(j)}(x+y)+(x+y+n-1)c_{n-1}^{(j)}(x+y)\bigr)\quad(k\ge 1)\,,\end{aligned} \tag{4}$$

$$\begin{aligned}T_2^{(-k)}(n;x,y)&=c_n^{(1)}(x+y-1)\\&\quad+n\sum_{j=0}^{k-1}\bigl(c_n^{(-j)}(x+y)+(x+y+n-1)c_{n-1}^{(-j)}(x+y)\bigr)\quad(k\ge 1)\,,\end{aligned} \tag{5}$$

where $c_n^{(1)}(x+y-1)=c_n(x+y)+nc_{n-1}(x+y)$ $(n\ge 1)$.

Proof.

$$\begin{aligned}\sum_{n=0}^\infty T_2^{(0)}(n;z_1,z_2)\frac{x^n}{n!}&=\frac{1}{(1+x)^{z_1+z_2}}\frac{x}{\ln(1+x)}(1+x)\\&=\frac{1}{(1+x)^{z_1+z_2-1}}\frac{x}{\ln(1+x)}\\&=\sum_{n=0}^\infty c_n^{(1)}(z_1+z_2-1)\frac{x^n}{n!}\,.\end{aligned}$$

Hence, the identity (3) holds.

Since

$$\sum_{n=0}^\infty c_n^{(k)}(z)\frac{x^n}{n!}=\frac{1}{(1+x)^z}\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}\,, \tag{6}$$

we have

$$\begin{aligned}\sum_{n=0}^\infty T_2^{(0)}(n;z_1,z_2)\frac{x^n}{n!}&=(1+x)\frac{1}{(1+x)^{z_1+z_2}}\frac{x}{\ln(1+x)}\\&=(1+x)\sum_{n=0}^\infty c_n(z_1+z_2)\frac{x^n}{n!}\\&=\sum_{n=0}^\infty\bigl(c_n(z_1+z_2)+nc_{n-1}(z_1+z_2)\bigr)\frac{x^n}{n!}\,.\end{aligned}$$

Next, since

$$
\begin{aligned}
\frac{d}{dx}\mathrm{Lif}_k(x)&=\frac{1}{x}\sum_{m=0}^\infty\frac{mx^m}{m!(m+1)^k}\\
&=\frac{1}{x}\sum_{m=0}^\infty\left(\frac{x^m}{m!(m+1)^{k-1}}-\frac{x^m}{m!(m+1)^k}\right)=\frac{\mathrm{Lif}_{k-1}(x)-\mathrm{Lif}_k(x)}{x}
\end{aligned}
$$

and $G_0(x)=1+x$, we have

$$
\begin{aligned}
\sum_{j=1}^k\frac{d}{dx}G_j(x)&=\sum_{j=1}^k\frac{G_{j-1}(x)-G_j(x)}{(1+x)\ln(1+x)}\\
&=\frac{1}{\ln(1+x)}-\frac{G_k(x)}{(1+x)\ln(1+x)}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{n=0}^\infty T_2^{(k)}(n;z_1,z_2)\frac{x^n}{n!}\\
&=\frac{1}{(1+x)^{z_1+z_2}}\frac{x}{\ln(1+x)}G_k(x)\\
&=\frac{1}{(1+x)^{z_1+z_2}}\frac{x(1+x)}{\ln(1+x)}-\frac{1}{(1+x)^{z_1+z_2}}x(1+x)\sum_{j=1}^k\frac{d}{dx}G_j(x)\\
&=\sum_{n=0}^\infty c_n^{(1)}(z_1+z_2-1)\frac{x^n}{n!}-\frac{1}{(1+x)^{z_1+z_2}}x(1+x)\sum_{j=1}^k\sum_{n=0}^\infty c_{n+1}^{(j)}\frac{x^n}{n!}\,.
\end{aligned}
$$

Since by (6)

$$
\begin{aligned}
\frac{x(1+x)}{(1+x)^z}\sum_{n=0}^\infty c_{n+1}^{(j)}\frac{x^n}{n!}&=x(1+x)\frac{d}{dx}\frac{1}{(1+x)^z}\sum_{n=0}^\infty c_n^{(j)}\frac{x^n}{n!}+\frac{zx}{(1+x)^z}\sum_{n=0}^\infty c_n^{(j)}\frac{x^n}{n!}\\
&=x(1+x)\frac{d}{dx}\sum_{n=0}^\infty c_n^{(j)}(z)\frac{x^n}{n!}+zx\sum_{n=0}^\infty c_n^{(j)}(z)\frac{x^n}{n!}\\
&=x(1+x)\sum_{n=0}^\infty c_{n+1}^{(j)}(z)\frac{x^n}{n!}+zx\sum_{n=0}^\infty c_n^{(j)}(z)\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty(nc_n^{(j)}(z)+n(n-1)c_{n-1}^{(j)}(z))\frac{x^n}{n!}+z\sum_{n=0}^\infty nc_{n-1}^{(j)}(z)\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty(nc_n^{(j)}(z)+n(z+n-1)c_{n-1}^{(j)}(z))\frac{x^n}{n!}\,.
\end{aligned}
$$

we have

$$
\begin{aligned}
&\sum_{n=0}^\infty T_2^{(k)}(n;z_1,z_2)\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty c_n^{(1)}(z_1+z_2-1)\frac{x^n}{n!}\\
&\qquad -\sum_{n=0}^\infty\sum_{j=1}^k\bigl(nc_n^{(j)}(z_1+z_2)+n(z_1+z_2+n-1)c_{n-1}^{(j)}(z_1+z_2)\bigr)\frac{x^n}{n!}\,.
\end{aligned}
$$

Therefore, we get the identity (4).

Finally, by

$$
\frac{d}{dx}\mathrm{Lif}_{-k}(x)=\frac{\mathrm{Lif}_{-k-1}(x)-\mathrm{Lif}_{-k}(x)}{x}\,,
$$

we have

$$
\begin{aligned}
\sum_{j=0}^{k-1}\frac{d}{dx}G_{-j}(x)&=\sum_{j=0}^{k-1}\frac{G_{-j-1}(x)-G_{-j}(x)}{(1+x)\ln(1+x)}\\
&=\frac{G_{-k}(x)}{(1+x)\ln(1+x)}-\frac{1}{\ln(1+x)}\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{n=0}^\infty T_2^{(-k)}(n;z_1,z_2)\frac{x^n}{n!}\\
&=\frac{1}{(1+x)^{z_1+z_2}}\frac{x}{\ln(1+x)}G_{-k}(x)\\
&=\frac{1}{(1+x)^{z_1+z_2}}\frac{x(1+x)}{\ln(1+x)}+\frac{1}{(1+x)^{z_1+z_2}}x(1+x)\sum_{j=0}^{k-1}\frac{d}{dx}G_{-j}(x)\\
&=\sum_{n=0}^\infty c_n^{(1)}(z_1+z_2-1)\frac{x^n}{n!}+\frac{1}{(1+x)^{z_1+z_2}}x(1+x)\sum_{j=0}^{k-1}\sum_{n=0}^\infty c_{n+1}^{(-j)}\frac{x^n}{n!}\\
&=\sum_{n=0}^\infty c_n^{(1)}(z_1+z_2-1)\frac{x^n}{n!}\\
&\qquad +\sum_{n=0}^\infty\sum_{j=0}^{k-1}\bigl(nc_n^{(-j)}(z_1+z_2)+n(z_1+z_2+n-1)c_{n-1}^{(-j)}(z_1+z_2)\bigr)\frac{x^n}{n!}\,.
\end{aligned}
$$

Therefore, we get the identity (5). □

Putting $k=1$ in (4), we have the identity in Theorem 1.1. This is also an analogous formula to Euler product (1).

3. Explicit formulae for $T_3^{(k)}(n;x,y,z)$

Theorem 3.1. *For $n\ge 0$ and $k\ge 1$ we have*

$$\begin{aligned}T_3^{(0)}(n;x,y,z)&=T_2^{(1)}(n;x+y,z-1)\\&=-n(x+y+z+n-3)c_{n-1}(x+y+z-1)\\&\quad-(n-1)c_n(x+y+z-1)\,,\end{aligned}\tag{7}$$

$$\begin{aligned}T_3^{(k)}(n;x,y,z)&=T_3^{(0)}(n;x,y,z)-(1-2^{-k})nT_2^{(0)}(n-1;x+y,z)\\&+n(n-1)\sum_{j=1}^k(1-2^{j-k-1})\\&\times\bigl(((x+y+z)^2+(n-2)(2(x+y+z)+n-2))c_{n-2}^{(j)}(x+y+z)\\&+(2(x+y+z)+2n-3)c_{n-1}^{(j)}(x+y+z)+c_n^{(j)}(x+y+z)\bigr)\,,\end{aligned}\tag{8}$$

$$\begin{aligned}T_3^{(-k)}(n;x,y,z)&=T_3^{(0)}(n;x,y,z)+(2^k-1)nT_2^{(0)}(n-1;x+y,z)\\&+n(n-1)\sum_{j=0}^{k-2}(2^{k-j-1}-1)\\&\times\bigl(((x+y+z)^2+(n-2)(2(x+y+z)+n-2))c_{n-2}^{(-j)}(x+y+z)\\&+(2(x+y+z)+2n-3)c_{n-1}^{(-j)}(x+y+z)+c_n^{(-j)}(x+y+z)\bigr)\,.\end{aligned}\tag{9}$$

Remark. *If $x=y=z=0$, then by $c_n(-1)=c_n+nc_{n-1}$, then the relations in Theorem 3.1 is reduced to relations (55), (56) and (57) in* [**12**, Theorem 6]. *Namely,*

$$\begin{aligned}T_3^{(0)}(n)&=T_2^{(1)}(n;0,-1)\\&=-n(n-3)c_{n-1}(-1)-(n-1)c_n(-1)\\&=-n(n-1)(n-3)c_{n-2}-2n(n-2)c_{n-1}-(n-1)c_n\,,\end{aligned}\tag{10}$$

(11)

$$\begin{aligned}T_3^{(k)}(n)&=T_3^{(0)}(n)-(1-2^{-k})n(c_{n-1}+(n-1)c_{n-2})\\&+n(n-1)\sum_{j=1}^k(1-2^{j-k-1})((n-2)^2c_{n-2}^{(j)}+(2n-3)c_{n-1}^{(j)}+c_n^{(j)})\quad(k\ge 1)\,,\end{aligned}$$

(12)

$$\begin{aligned}T_3^{(-k)}(n)&=T_3^{(0)}(n)+(2^k-1)n(c_{n-1}+(n-1)c_{n-2})\\&+n(n-1)\sum_{j=0}^{k-2}(2^{k-j-1}-1)\bigl((n-2)^2c_{n-2}^{(-j)}+(2n-3)c_{n-1}^{(-j)}+c_n^{(-j)}\bigr)\quad(k\ge 1)\,.\end{aligned}$$

Proof.

$$\begin{aligned}\sum_{n=0}^\infty T_3^{(0)}(n;z_1,z_2,z_3)\frac{x^n}{n!}&=\frac{1}{(1+x)^{z_1+z_2+z_3}}\left(\frac{x}{\ln(1+x)}\right)^2(1+x)\\&=\frac{1}{(1+x)^{z_1+z_2+z_3-1}}\left(\frac{x}{\ln(1+x)}\right)\frac{x}{\ln(1+x)}\\&=\sum_{n=0}^\infty T_2^{(1)}(n;z_1+z_2,z_3-1)\frac{x^n}{n!}\,.\end{aligned}$$

Hence, the identities (7) and (10) with $x=y=z=0$ hold.

By Lemma 1.3 with $m=2$

$$\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_k(x)=\frac{(2G_k(x)-G_{k-1}(x))-(2G_{k-1}(x)-G_{k-2}(x))}{(1+x)^2(\ln(1+x))^2}\,.$$

Thus,

$$\sum_{j=1}^l\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_j(x)=\frac{2G_l(x)-G_{l-1}(x)}{(1+x)^2(\ln(1+x))^2}-\frac{2G_0(x)-G_{-1}(x)}{(1+x)^2(\ln(1+x))^2}\,.$$

By multiplying both sides by 2^{l-1} and summing over l from 1 to k, we obtain

$$\begin{aligned}2^k\frac{G_k(x)}{(1+x)^2(\ln(1+x))^2}&=\frac{G_0(x)}{(1+x)^2(\ln(1+x))^2}+\left(\sum_{l=1}^k 2^{l-1}\right)\frac{2G_0(x)-G_{-1}(x)}{(1+x)^2(\ln(1+x))^2}\\&\quad+\sum_{l=1}^k 2^{l-1}\sum_{j=1}^l\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_j(x)\\&=\frac{(2^{k+1}-1)G_0(x)-(2^k-1)G_{-1}(x)}{(1+x)^2(\ln(1+x))^2}\\&\quad+\sum_{j=1}^k\left(\sum_{l=j}^k 2^{l-1}\right)\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_j(x)\,.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{2^k}{(1+x)^z}\left(\frac{x}{\ln(1+x)}\right)^2G_k(x)\\&=\frac{2^{k+1}-1}{(1+x)^z}\frac{(1+x)x^2}{(\ln(1+x))^2}-\frac{2^k-1}{(1+x)^z}\frac{(1+x)x^2(\ln(1+x)+1)}{(\ln(1+x))^2}\\&\quad+\frac{x^2(1+x)^2}{(1+x)^z}\sum_{j=1}^k(2^k-2^{j-1})\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_j(x)\,,\end{aligned}$$

where $z = z_1 + z_2 + z_3$. By Lemma 1.4

$$\begin{aligned}
&\frac{x^2(1+x)^2}{(1+x)^z}\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_j(x)\\
&= x^2\sum_{n=0}^{\infty}\big(z(z-1)c_n^{(j)}(z)+2(1+x)zc_{n+1}^{(j)}(z)+(1+x)^2c_{n+2}^{(j)}(z)\big)\frac{x^n}{n!}\\
&\quad + x^2\sum_{n=0}^{\infty}\big(zc_n^{(j)}(z)+(1+x)c_{n+1}^{(j)}(z)\big)\frac{x^n}{n!}\\
&= \sum_{n=0}^{\infty} n(n-1)\big((z^2+(n-2)(2z+n-2))c_{n-2}^{(j)}(z)\\
&\qquad\qquad + (2z+2n-3)c_{n-1}^{(j)}(z)+c_n^{(j)}(z)\big)\frac{x^n}{n!}\,.
\end{aligned}$$

By comparing the coefficients of $x^n/n!$ in both sides,

$$\begin{aligned}
&2^kT_3^{(k)}(n;z_1,z_2,z_3)\\
&= (2^{k+1}-1)T_3^{(0)}(n;z_1,z_2,z_3)-(2^k-1)nT_2^{(0)}(n-1;z_1+z_2,z_3)\\
&\qquad - (2^k-1)T_3^{(0)}(n;z_1,z_2,z_3)\\
&\quad + n(n-1)\sum_{j=1}^{k}(2^k-2^{j-1})\big((z^2+(n-2)(2z+n-2))c_{n-2}^{(j)}(z)\\
&\qquad + (2z+2n-3)c_{n-1}^{(j)}(z)+c_n^{(j)}(z)\big)\\
&= 2^kT_3^{(0)}(n;z_1,z_2,z_3)-(2^k-1)nT_2^{(0)}(n-1;z_1+z_2,z_3)\\
&\quad + n(n-1)\sum_{j=1}^{k}(2^k-2^{j-1})\big((z^2+(n-2)(2z+n-2))c_{n-2}^{(j)}(z)\\
&\qquad + (2z+2n-3)c_{n-1}^{(j)}(z)+c_n^{(j)}(z)\big)\,.
\end{aligned}$$

Dividing both sides by 2^k, we have the identities (8) and (11) with $x = y = z = 0$.

Finally, since

$$\begin{aligned}
&\sum_{j=0}^{l-2}\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_{-j}(x)\\
&= \sum_{j=0}^{l-2}\frac{(2G_{-j}(x)-G_{-j-1}(x))-(2G_{-j-1}(x)-G_{-j-2}(x))}{(1+x)^2(\ln(1+x))^2}\\
&= \sum_{j=0}^{l-2}\left(\frac{2G_0(x)-G_{-1}(x)}{(1+x)^2(\ln(1+x))^2}-\frac{2G_{-l+1}(x)-G_{-l}(x)}{(1+x)^2(\ln(1+x))^2}\right)\,,
\end{aligned}$$

by multiplying both sides by 2^{-l} and summing over l from 2 to k, we obtain

$$2^{-k}\frac{G_{-k}(x)}{(1+x)^2(\ln(1+x))^2}=\frac{(1-2^{-k})G_{-1}(x)-(1-2^{-k+1})G_0(x)}{(1+x)^2(\ln(1+x))^2}+\sum_{j=0}^{k-2}\left(\sum_{l=j+2}^{k}2^{-l}\right)\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_{-j}(x)\,.$$

Hence, we have

$$\begin{aligned}&\frac{2^{-k}}{(1+x)^z}\left(\frac{x}{\ln(1+x)}\right)^2 G_{-k}(x)\\&=\frac{1-2^{-k}}{(1+x)^z}\frac{(1+x)x^2}{\ln(1+x)}+\frac{2^{-k}}{(1+x)^z}\frac{(1+x)x^2}{(\ln(1+x))^2}\\&\quad+\frac{x^2(1+x)^2}{(1+x)^z}\sum_{j=0}^{k-2}(2^{-j-1}-2^{-k})\left(\frac{d^2}{dx^2}+\frac{1}{1+x}\frac{d}{dx}\right)G_{-j}(x)\,,\end{aligned}$$

yielding the identities (9) and (12) with $x=y=z=0$. □

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